On the occurrence of condensations in steady axisymmetric jets

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In experiments on the stability of a submerged axisymmetric jet at Reynolds numbers large compared with unity, Reynolds (1962) observed axisymmetric 'condensations' which appear to grow spontaneously, whereas Batchelor & Gill (1962) have shown that infinitesimal disturbances of this type do not grow in an inviscid fluid. Here it is shown that axisymmetric disturbances do not grow in a slightly viscous fluid either, and the solution for which the rate of damping is smallest is found. It is suggested that the growth of small but finite disturbances is responsible for the condensations observed. The order of magnitude of the disturbance velocity at which non-linear effects could produce growth of a disturbance is found to depend on the wave-number of the disturbance. The smallest velocity which a 'finite' disturbance may have is found to be of order $R^{-\frac{2}{3}}$, and corresponds to a disturbance whose wave-number is of order $R^{\frac{1}{3}}$, R being the Reynolds number based on the local radius and maximum velocity of the jet. On the assumption that some disturbances whose velocity is of this order will grow, deductions are made as to the size, position, wave-number, and point of appearance of condensations. The deductions appear to agree with the experimental results.

1. Comparison of infinitesimal theory with experiment

The quantities used in the next two sections will be assumed to be made non-dimensional by choosing length and time scales so that the radius of the jet and the velocity of the jet at the centre are both unity. Only axisymmetric disturbances are considered. The paper by Batchelor & Gill (1962) will be referred to as paper I and the paper by Reynolds (1962) as paper II.

(i) *Experiment.* Condensations (see II) are seen to appear, apparently spontaneously, over a range of Reynolds numbers R of about 50 to 250, while the (non-dimensional) wave-number α is round about 5 for all these Reynolds numbers. Thus both R and αR are large, and the conditions (I, § 1) required for the validity of parallel flow theory are satisfied. The condensations appear at the centre of the jet, and their radius appears to be small compared with the width r_0 of the jet, as calculated by equation (1.9) of I.

(ii) *Theory.* It is shown in I that, in inviscid fluid, infinitesimal (axisymmetric) disturbances *do not grow* (§ 4), and that the rate of damping is non-zero, except possibly when the wave-speed of the disturbance is equal to the velocity of the jet at the centre (§ 5). It is also shown that if such neutral solutions exist, they have a singularity at the centre (except for the trivial case of zero wave-number). Thus, in the limit of vanishing viscosity, the amplification rate tends to a non-zero negative value except possibly for disturbances whose wave-speed is unity. This raises two questions. First, do neutral inviscid solutions exist at all? And secondly, if they do exist, does the amplification rate tend to zero through positive values (as it does for disturbances to the boundary layer on a flat plate) or through negative values? In the absence of rigid boundaries, it seems almost certain that viscosity will have a stabilizing effect. This is confirmed in the next section, where it is shown that solutions neutral in the inviscid limit do exist and that they are stable. Their form and rate of damping are found explicitly.

(iii) *Comparison.* The fact that condensations appear at all disagrees with the predictions of the infinitesimal theory, so presumably the appearance of these condensations is due to the growth of small, but finite, disturbances. Now if small finite disturbances of a certain wave-number grow, one would expect that infinitesimal disturbances of the same wave-number are only weakly damped. This points to the importance of solutions which are neutral in the limit of vanishing viscosity. The next section is devoted to finding these solutions.

2. Solutions neutral in the inviscid limit

The full governing equation

The equation is most readily handled in terms of the Stokes stream function Ψ such that the axial and radial components (u, v) of the velocity are given by

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \Psi}{\partial x}, \tag{2.1}$$

where (x, r, ϕ) are cylindrical polar co-ordinates. The azimuthal component of vorticity is then

$$rZ = -\frac{1}{r}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\Psi \tag{2.2}$$

and the Navier–Stokes equation has the non-dimensional form

$$\frac{\partial Z}{\partial t} + \frac{1}{r}\frac{\partial \Psi}{\partial r}\frac{\partial Z}{\partial x} - \frac{1}{r}\frac{\partial Z}{\partial r}\frac{\partial \Psi}{\partial x} = \frac{1}{R}\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right) Z. \tag{2.3}$$

Here R is the Reynolds number based on the radius and maximum velocity of the jet. The primary flow is in the axial direction with velocity $U(r)$, so has stream function $\Psi = \int rU(r)\,dr$ and azimuthal vorticity component $rZ = -dU/dr$. Following the usual practice (I, 4.16 with $n = 0$), we consider a disturbance whose stream function has the form

$$\mathscr{R}\{\phi(r)\, e^{i\alpha(x-ct)}\}. \tag{2.4}$$

The disturbance azimuthal vorticity is therefore $\mathscr{R}\{r\zeta(r)\, e^{i\alpha(x-ct)}\}$, where, by (2.2),

$$\zeta = -\frac{1}{r^2}\left(\phi'' - \frac{\phi'}{r} - \alpha^2\phi\right), \tag{2.5}$$

the prime denoting differentiation with respect to r. The linearized equation corresponding to infinitesimal disturbances is obtained in the usual way by substituting the total stream function, $\Psi = \int rU\,dr + \mathscr{R}\{\phi(r)\,e^{i\alpha(x-ct)}\}$, in the Navier–Stokes equation (2.3), and neglecting squares of ϕ. It therefore has the form

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(\zeta e^{i\alpha(x-ct)})+\frac{1}{r}\left(\frac{U'}{r}\right)'\frac{\partial}{\partial x}(\phi e^{i\alpha(x-ct)}) = \frac{1}{R}\left(\frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}\right)(\zeta e^{i\alpha(x-ct)}),$$

that is

$$(U-c)\,\zeta+\frac{1}{r}\left(\frac{U'}{r}\right)'\phi = -\frac{i}{\alpha R}\left(\zeta''+\frac{3}{r}\zeta'-\alpha^2\zeta\right). \tag{2.6}$$

In terms of the single dependent variable, ϕ, this equation is, by (2.5), the fourth-order equation (cf. Sexl 1927)

$$(U-c)\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-\alpha^2\right)\phi-r\left(\frac{U'}{r}\right)'\phi = -\frac{i}{\alpha R}\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-\alpha^2\right)^2\phi. \tag{2.7}$$

The boundary conditions, as in paper I, are

(*a*) that the disturbance velocity components u and v vanish as r approaches ∞, i.e.

$$\frac{1}{r}\frac{\phi'}{r},\quad \frac{\alpha\phi}{r}\to 0 \quad\text{as}\quad r\to\infty, \tag{2.8}$$

and (*b*) the regularity conditions that u is finite and v zero at $r = 0$, i.e.

$$\frac{1}{r}\frac{\phi'}{r}\ \text{finite},\quad\text{and}\quad \frac{\alpha\phi}{r} = 0 \quad\text{at}\quad r = 0. \tag{2.9}$$

Singular solutions of the inviscid equation

The inviscid equation is obtained by formally putting $1/\alpha R$ equal to zero in (2.7) and is the same as the governing equation (I, 2.16 with $n = 0$; $\phi = -rG$ by (2.4) above and I, 4.16) of paper I. Note, however, that when the wave speed c is the same as the velocity $U(0)$ at the centre of the jet, the left-hand side of (2.7) vanishes, and the deduction of the inviscid equation as a limit of the full equation (2.7) is no longer valid. In §5 of paper I it was shown that when the wave-speed is equal to $U(0)$, the inviscid equation does not in general have a solution satisfying both (2.8) and (2.9), so that the only solutions which can exist are singular at $r = 0$. The radial velocity of the disturbance was shown in the same section to behave like $1/r$ near $r = 0$, so that the function ϕ, by (2.1) and (2.4) tends to a non-zero value as r tends to zero. The boundary condition (2.9), however, requires that ϕ be zero at $r = 0$, so that ϕ must have a discontinuity at $r = 0$.

In particular, for the velocity profile

$$U = (1+r^2)^{-2}$$

appropriate to the experimental jet (see I, 6.1), and the wave-speed equal to $U(0)$, that is $c = 1$, the inviscid equation is

$$\phi''-\frac{\phi'}{r}-\alpha^2\phi+\frac{24}{(2+r^2)(1+r^2)^2}\phi = 0. \tag{2.10}$$

When the wave-number α is large, the last term on the left-hand side of (2.10) can be neglected, and the approximate solution which vanishes as r tends to infinity is

$$\phi = \begin{cases} \alpha r K_1(\alpha r) & \text{for} \quad r > 0 \\ 0 & \text{for} \quad r = 0 \end{cases}. \tag{2.11}$$

It now remains to be shown that such a singular solution does represent a disturbance in a viscous fluid in the limit of vanishing viscosity. When the viscosity is small but non-zero, the singular line $r = 0$ on which ϕ is discontinuous is replaced by a narrow region near the centre of the jet over which ϕ changes rapidly. This 'critical layer' shrinks to the line $r = 0$ when αR tends to infinity. The problem is to find under what conditions ϕ can change in the critical layer in such a way as to balance the discontinuity in the inviscid solution (2.11), so that the overall solution is continuous and satisfies the boundary conditions.

Solution in the critical layer

The method of finding the solution in the critical layer is much the same as the method employed for plane unidirectional flows (see, for example, Lin, 1955, §3.6 and chapter 8). The layer is maintained as a balance between the diffusion of vorticity by viscosity (right-hand side of (2.6), of order $\zeta/\alpha R r^2$) and the convection of vorticity by the primary flow, $(U-1)$, relative to the centre of the jet (left-hand side of (2.6) of order $r^2\zeta$). The radius of the layer is, therefore, of order $(\alpha R)^{-\frac{1}{4}}$. Since by (2.5) ϕ is of order $r^4\zeta$ (or smaller if α is very large), the second term on the left-hand side of (2.6) is small compared with the first, and so may be neglected. Finally, the wave speed, c, is unity to the first order as αR tends to infinity, but in the critical layer it is necessary to make a second-order correction. This correction term is vital since it determines the damping of the infinitesimal disturbance, and will be of the same order, $(\alpha R)^{-\frac{1}{2}}$, as the velocity of the primary flow in the critical layer, relative to the centre of the jet. To absorb the coefficient $i(\alpha/R)$ of ζ on the right-hand side of (2.6) we define the correction term c_1 thus

$$c = 1 - i\frac{\alpha}{R} - (\alpha R)^{-\frac{1}{2}} c_1. \tag{2.12}$$

Substituting (2.12) in (2.6) and making these approximations for r small, we have

$$(-2r^2 + (\alpha R)^{-\frac{1}{2}} c_1)\,\zeta = -i(\alpha R)^{-1}(\zeta'' + (3/r)\,\zeta'). \tag{2.13}$$

We make two observations about the solution ζ of this equation:

(i) ζ is an even function of r, so depends on r^2 rather than r.

(ii) the solution bounded as $(\alpha R)^{\frac{1}{4}} r$ tends to infinity behaves, for large $(\alpha R)^{\frac{1}{4}} r$, like

$$\zeta \sim \exp\left[-\tfrac{1}{2}(1-i)(\alpha R)^{\frac{1}{2}} r^2\right].$$

This suggests the following changes of variable

$$y = (1-i)(\alpha R)^{\frac{1}{2}} r^2, \tag{2.14}$$

$$\zeta = \eta \exp\left[-\tfrac{1}{2}(1-i)(\alpha R)^{\frac{1}{2}} r^2\right] = \eta\, e^{-\frac{1}{2}y}, \tag{2.15}$$

which puts (2.13) in the form

$$y\frac{d^2\eta}{dy^2} + (2-y)\frac{d\eta}{dy} - (1 - \tfrac{1}{8}(1-i)\,c_1)\,\eta = 0. \tag{2.16}$$

The problem has now become a classical eigenvalue problem: to find the values of c_1 such that the solution, η, of (2.16) is regular at $r = 0$ and such that $\zeta = \eta\, e^{-\frac{1}{2}y}$ vanishes as y becomes large. The eigenvalues are given by

$$\tfrac{1}{8}(1-i)\,c_1 = N \quad (N = 1, 2, 3, \ldots), \tag{2.17}$$

and the eigenfunctions are Laguerre polynomials (see Erdélyi, 1953, vol. II, §10.12).

Substituting (2.17) in (2.12),

$$c = 1 - i\alpha/R - 4N(\alpha R)^{-\frac{1}{2}}(1+i), \tag{2.18}$$

so that the damping is given by

$$c_i = -\frac{\alpha}{R} - 4N(\alpha R)^{-\frac{1}{2}}. \tag{2.19}$$

We see that the least damped mode is given by $N = 1$, when the eigenfunction is simply η = constant = B, say, or

$$\zeta = B\,e^{-\frac{1}{2}y}. \tag{2.20}$$

To find the corresponding stream function, ϕ, we rewrite (2.5) in terms of y, that is

$$\frac{d^2\phi}{dy^2} - \frac{(1+i)}{8}\left(\frac{\alpha^3}{R}\right)^{\frac{1}{2}}\frac{\phi}{y} = -\frac{i\zeta}{8\alpha R} = -\frac{iB}{8\alpha R}e^{-\frac{1}{2}y}, \tag{2.21}$$

which can be integrated to obtain a stream function, ϕ, which satisfies the boundary conditions. The behaviour of ϕ depends on whether $\alpha/R^{\frac{1}{3}}$ is large, small or of order unity, as this determines which term on the left-hand side of (2.21) is the more important in the critical layer. For instance, if the wavelength of the disturbance is large compared with the radius of the critical layer ($\alpha \ll (\alpha R)^{\frac{1}{4}}$, or $\alpha \ll R^{\frac{1}{3}}$) the second term on the left-hand side of (2.21) may be neglected in the critical layer, so that the first approximation to ϕ in the critical layer which satisfies the boundary condition (2.9) is of the form

$$\phi_0 = \frac{iB}{2\alpha R}(1 - e^{-\frac{1}{2}y}) + Dy, \tag{2.22}$$

D being a constant of integration. If the wavelength is at the same time small compared with the radius of the jet (i.e. $\alpha \gg 1$) the solution outside the critical layer is given by the first half of the inviscid solution (2.11). The constants B and D can then be calculated by the usual matching procedure, i.e. by comparing (2.11) and (2.22) at a value of r which is at the same time large compared with $(\alpha R)^{\frac{1}{4}}$ (i.e. y large) and small compared with $1/\alpha$ (i.e. αr small). This gives $B = -2i\alpha R$, $D = 0$ to the first order, and so

$$\phi \sim \begin{cases} 1 - \exp\left[-\tfrac{1}{2}(1-i)\,(\alpha R)^{\frac{1}{2}} r^2\right] & \text{in the critical layer,} \\ \alpha r K_1(\alpha r) & \text{outside the critical layer.} \end{cases} \tag{2.23}$$

To the first order, the disturbance velocity is, by (2.1) and (2.4), only important in the critical layer, where it is in the axial direction and given by

$$\begin{aligned} u &= \mathscr{R}\{(1-i)\,(\alpha R)^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}(1-i)\,(\alpha R)^{\frac{1}{2}} r^2 + i\alpha(x-t)\right]\} \\ &= (2\alpha R)^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}(\alpha R)^{\frac{1}{2}} r^2\right] \cos\left[\alpha(x-t) + \tfrac{1}{2}(\alpha R)^{\frac{1}{2}} r^2 - \tfrac{1}{4}\pi\right], \end{aligned}$$

that is, the disturbance velocity amplitude has a Gaussian distribution with maximum at the centre of the jet, and the phase of u changes significantly across the layer. To calculate the disturbance velocity to a stage where it is significant outside the critical layer, the second term ϕ_1 in the series

$$\phi = \phi_0 + (\alpha^3/R)^{\frac{1}{2}}\phi_1 + (\alpha^3/R)\phi_2 + \dots$$

for the stream function in the critical layer is necessary. The calculation of ϕ_1 is straightforward.

On the other hand, if the wavelength of the disturbance is small compared with the radius of the critical layer ($\alpha \gg R^{\frac{1}{3}}$) it is seen from (2.21) that the stream function in the critical layer is, to the first order, of the form

$$\phi \sim y\,e^{-\frac{1}{2}y} = \text{constant} \times r^2 \exp\left[-\tfrac{1}{2}(\alpha R)^{\frac{1}{2}}(1-i)\,r^2\right],$$

and is exponentially small outside the layer. From (2.19), the damping is now dominated by the first term, $-\alpha/R$, which came from the viscous term in the equation of motion, and increases without limit as α increases. This means that for very large wave-numbers, the energy exchange processes are dominated by the dissipation of energy by viscosity in the critical layer.

To sum up: there exist singular solutions of the inviscid equation (2.10) which correspond to infinitesimal axisymmetric disturbances in a slightly viscous fluid (or rather for a situation in which the parameter αR is large). For large wave-numbers, they have the following properties:

(i) The disturbances are localized near the centre of the jet.

(ii) The radius of the region outside which disturbance velocities vanish is small compared with the radius of the jet. The ratio of these radii is $(\alpha R)^{-\frac{1}{4}}$.

(iii) The wave velocity of the disturbance is a little less than the maximum velocity of the jet.

(iv) The rate of damping is small, unless the wave-number is very large. [By (2.19), $-\alpha c_i$ is of order unity when α is of order $R^{\frac{1}{2}}$, and increases without limit as the wave-number increases further.]

The condensations observed in experiments have the first two properties (as far as one can judge from the experimental data, the ratio of disturbance radius to jet radius and $(\alpha R)^{-\frac{1}{4}}$ are about the same value), and no measurements of the speed of the condensations are available for comparison with (iii). However, there is the important difference that the experimental disturbances seem to appear spontaneously, and so presumably are not damped like the infinitesimal disturbances. This leads us to consider finite disturbances similar to the infinitesimal disturbances found above.

3. Finite disturbances

Criterion for 'finiteness'

The loss of energy by infinitesimal disturbances of the kind considered in §2 is the resultant of two processes: the transfer of energy between the mean flow and the disturbance, and the dissipation of energy by viscosity. Both processes take place in the critical layer: the transfer between mean flow and disturbance since the Reynolds stress is zero outside the layer (ϕ real), and the dissipation by viscosity since the flow outside the layer is inviscid. This energy relationship will be upset once the disturbance velocity in the critical layer becomes comparable with the primary flow velocity relative to the centre of the jet $(U-1)$, for then in the critical layer convection of vorticity by the disturbance will be as important as convection by the primary flow, or, more precisely, the non-linear terms

$$\frac{1}{r}\frac{\partial \Psi}{\partial r}\frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{1}{r}\frac{\partial Z}{\partial r}\frac{\partial \Psi}{\partial x}$$

in (2.3), where here Z and Ψ refer to the disturbance, then become of the same order as the linear terms

$$(U-1)\frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{1}{R}\left(\frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r}\right)Z.$$

Thus a finite disturbance is one whose velocity in the critical layer is the same order, $(\alpha R)^{-\frac{1}{2}}$, as $U-1$, for disturbances whose velocities are of smaller order will decay like infinitesimal disturbances, whereas disturbances of the same or greater order will not. For a disturbance to grow, then, its velocity must at least be of order $(\alpha R)^{-\frac{1}{2}}$. To make possible deductions about the size, position and wavelength of the disturbances that are most likely to grow, we postulate that *some* disturbances of just this order, $(\alpha R)^{-\frac{1}{2}}$, do grow. Notice that this order decreases as the wavelength decreases.

However, when the wavelength of the disturbance becomes so short that it is small compared with the radius of the critical layer (i.e. $\alpha \gg R^{\frac{1}{3}}$), disturbances with velocity of order $(\alpha R)^{-\frac{1}{2}}$ will no longer be as significant, since in (2.3) the diffusion term $(1/R)(\partial^2 Z/\partial x^2)$, which is of order $(\alpha^2/R)Z$, will be more important than the convection term $(U-1)(\partial Z/\partial x)$ associated with the primary flow, the latter being only of order $(\alpha/R)^{\frac{1}{2}} Z$. In fact the energy relationship appropriate to infinitesimal disturbances will not be changed significantly until the convection terms

$$\frac{1}{r}\frac{\partial \Psi}{\partial r}\frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{1}{r}\frac{\partial Z}{\partial r}\frac{\partial \Psi}{\partial x}$$

in (2.3) are comparable in the critical layer with the diffusion term $(1/R)(\partial^2 Z/\partial x^2)$, that is, until Ψ is of order $(\alpha/R)(\alpha R)^{-\frac{1}{2}} = \alpha^{\frac{1}{2}}R^{-\frac{3}{2}}$ and the velocity is of order $(\alpha/r)\Psi \sim \alpha^{\frac{7}{4}}R^{-\frac{5}{4}}$ (which increases with α). Even when disturbances are of this order, it seems doubtful if they would grow since convection by the primary flow is so dominated by viscous diffusion in the critical layer that one would expect dissipation of energy by viscosity to be far more important than the transfer of energy between the mean flow and the disturbance. Thus, for these very short wavelengths, 'finite' disturbances have velocity at least of order $\alpha^{\frac{7}{4}}R^{-\frac{5}{4}}$, but it seems that they will decay. In any case, 'finite' disturbances have their least order of magnitude when their wavelength is of the same order as the radius of the critical layer, that is when $\alpha \sim R^{\frac{1}{3}}$ and $(\alpha R)^{-\frac{1}{2}} = \alpha^{\frac{7}{4}}R^{-\frac{5}{4}} = R^{-\frac{2}{3}}$, in which case all the terms in (2.3) are of the same order in the critical layer. According to our postulate, some disturbances whose velocity in the critical layer is of order $R^{-\frac{2}{3}}$ will grow, so that the disturbances which are most likely to appear will have wave-number of order $R^{\frac{1}{3}}$ and radius of order $(1/\alpha) = (\alpha R)^{-\frac{1}{4}} = R^{-\frac{1}{3}}$.

We now ask ourselves in what form the disturbances are most likely to appear. For a disturbance whose velocity is just large enough to be regarded as finite, the diffusion terms on the right-hand side of (2.3) will be of the same order as the non-linear terms on the left-hand side; but, as the disturbance grows, we may expect the non-linear terms which depend on the square of the disturbance velocity to become progressively more important than the diffusion terms. Since only a finite amount of energy is available, we cannot expect the disturbance to grow indefinitely, so unless some secondary instability occurs, we can expect an equilibrium to be reached, which will be an inviscid rotational flow, steady relative to the centre of the disturbance.

Appearance of condensations at a certain point

One of the features of the condensations observed experimentally is that they do not appear until at a distance from the nozzle which increases with the Reynolds number. Any hypothesis which explains the presence of condensations should give some indication as to why their appearance is delayed, and to predict their wavelength. It seems very likely that the reason for the delayed appearance of the condensations is bound up with the fact that the velocity of the centre of the jet decreases with distance x from the nozzle according to the formula

$$U_0 = \frac{\nu R^2}{8x} \tag{3.1}$$

(by I, 1.10), while the radius of the jet increases according to the formula

$$r_0 = \frac{8x}{R} \tag{3.2}$$

(by I, 1.9). Now, according to our hypothesis, the minimum velocity of the disturbance which is most likely to appear, has order of magnitude

$$R^{-\frac{2}{3}} U_0 = \nu R^{\frac{4}{3}}/(8x),$$

and this quantity does not fall to a given value until a distance from the nozzle of order $R^{\frac{4}{3}}$. Thus if there is a background disturbance of a given level throughout the fluid, condensations resulting from the growth of such disturbances cannot be expected to appear until a distance from the nozzle proportional to the $\frac{4}{3}$-power of the Reynolds number. The wavelength of the disturbance which appears will be of the same order as the radius of the critical layer, that is, α will be of order $R^{\frac{1}{3}}$, and the radius of the disturbance compared with the radius of the jet will be of order $R^{-\frac{1}{3}}$.

Now the observed condensations appear over a range of Reynolds numbers of about 50 to 250, so $R^{\frac{1}{3}}$ varies from about 4 to 6. This agrees well with the observed value of α and the observed ratio of jet radius to the radius of the condensations. Also, a curve $x = \text{constant} \times R^{\frac{4}{3}}$, can be fitted quite well to the experimental curve relating the distance x from the nozzle at which condensations appear, to the Reynolds number, R. Some care must be taken here since the Reynolds number, R, based on the local radius and maximum velocity of the jet, may differ from the experimental Reynolds number, Re, which is based on the volume efflux and radius at the nozzle. In fact, it is shown in paper I (I, (1.6) to (1.8)) that these Reynolds numbers are the same if the flow in the nozzle is uniform. Andrade & Tsien (1937) show that due to the contraction in the exit tube, the flow in the nozzle is parabolic only for small Reynolds numbers, and is nearly uniform at large Reynolds numbers. In figure 1, the two curves

$$x = b\,Re^{\frac{4}{3}} \quad \text{and} \quad x = b(\tfrac{1}{2}\sqrt{3}\,Re)^{\frac{4}{3}}$$

are drawn, where b is a constant chosen to give the best fit. The experimental points are taken from paper II, and are the ones which, so the author learns from Dr Reynolds, mark the points where condensations first appeared. A change from parabolic to uniform nozzle flow over the range of Reynolds number suggested in the figure is not inconsistent with a rough estimate ($\frac{1}{4}$ in.) of the nozzle length supplied by the experimenter.

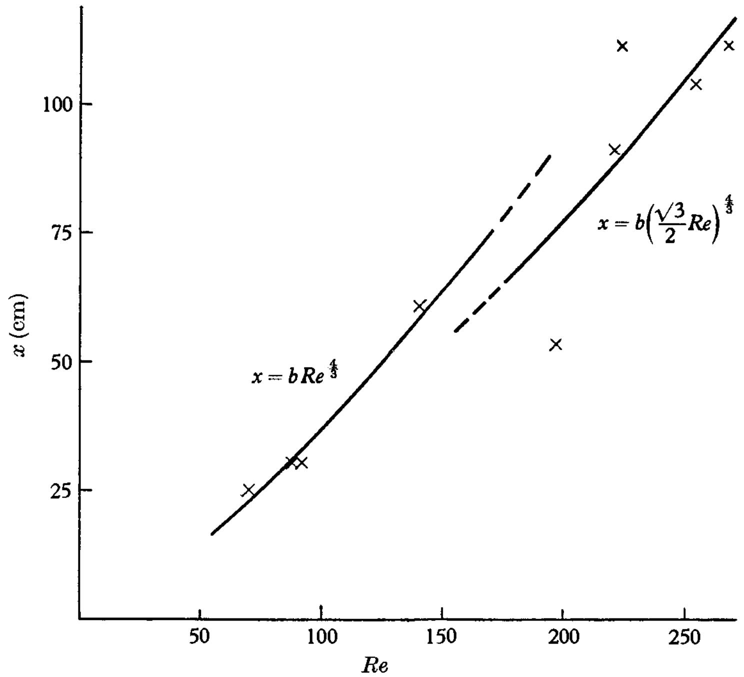

FIGURE 1. Variation of the length of undisturbed jet x with Reynolds number Re based on volume efflux and radius of the nozzle. The solid lines are theoretical curves and the experimental points are from Reynolds (1962).

The above explanation assumes the presence of background disturbances throughout the fluid, perhaps produced by vibration of the vessel. On the other hand, one might expect disturbances produced at or near the nozzle to play an important part. Such a disturbance would be convected downstream as a disturbance of a fixed frequency, β, so its wave-number $\alpha/r_0 = \beta/U_0$ would vary with distance downstream according to the formula

$$\alpha = \frac{(8x)^2 \beta}{R^3 \nu}$$

by (3.1) and (3.2). It can be shown that the velocity of an infinitesimal disturbance would be reduced by a factor

$$\exp \int_0^x \frac{\alpha c_i}{r_0}\, dx = \exp\left(-4N\sqrt{\left(\frac{\beta}{\nu}\right)\frac{x}{R}}\right) = \exp\left(-\frac{N}{2}\sqrt{(\alpha R)}\right),$$

which is a very large reduction. Although (2.19) will not hold near the nozzle because the parameter αR is not large there, it should be a good approximation over most of the distance, so that disturbances produced at the nozzle will be damped by such a factor that they would play no part in the formation of condensations.

A possible analytical procedure

In looking for a solution which corresponds to finite growing disturbances, a natural suggestion is try a stream function in the form of a Fourier series in x with coefficients which are series in the amplitude $A(t)$, that is of the type used by Stuart (1960) and Watson (1960, p. 376). This solution does not satisfy initial or boundary conditions which one would associate with the experimental conditions, but it does seem the easiest way to make a start. If the amplitude is defined so that its magnitude squared is equal to the (time-average) disturbance energy E per unit axial length, one obtains a differential equation for E which, to the first order in R and for all large α, has the form

$$\frac{1}{E}\frac{dE}{dt} = R^{-\frac{1}{3}} \sum_{m=0}^{\infty} R^{2m} g_m\left(\frac{\alpha^3}{R}\right) E^m. \tag{3.3}$$

To be consistent with infinitesimal theory in the limit as $E \to 0$, g_0 must be given by

$$g_0 = R^{\frac{1}{3}}\, 2\alpha c_i = -2\left(\frac{\alpha^3}{R}\right)^{\frac{2}{3}} - 8\left(\frac{\alpha^3}{R}\right)^{\frac{1}{6}},$$

by (2.16) for the case $N = 1$ where the infinitesimal disturbance is damped the least. For large α^3/R it is found that g_m is of order $(\alpha^3/R)^{\frac{2}{3}-m}$ and, for small α^3/R, g_m is of order $(\alpha^3/R)^{\frac{1}{6}+\frac{1}{2}m}$. The expansion (3.3) can be expected to be valid as long as E is small compared with the 'threshold' energy at which disturbances become finite. Since E amounts to an integral of the square of disturbance velocity over the cross-section of the critical layer, the order of magnitude of this threshold energy is, according to the criterion established in the first part of this section, $(\alpha R)^{-1}(\alpha R)^{-\frac{1}{2}} = (\alpha R)^{-\frac{3}{2}}$ when $\alpha \ll R^{\frac{1}{3}}$ and $\alpha^{\frac{7}{2}}R^{-\frac{5}{2}}(\alpha R)^{-\frac{1}{2}} = (\alpha/R)^3$ when $\alpha \gg R^{\frac{1}{3}}$, and so is R^{-2} when $\alpha \sim R^{\frac{1}{3}}$. These are just the orders of magnitude of E which make all the terms on the right-hand side of (3.3) of the same order.

A systematic way of calculating the coefficients g_m follows naturally from the assumed form of expansion and the definition of $|A|^2$, but it turns out that the only case where the first few coefficients can be worked out without a lengthy computing program is the case of least interest, namely the case where α^3/R is large. The first few terms in the asymptotic expansion of g_1 for large α^3/R are found to be

$$g_1 = -\frac{1}{2\pi}\left(\frac{\alpha^3}{R}\right)^{-\frac{1}{3}}\left[\frac{3}{4} - \frac{1}{2}\left(\frac{R}{\alpha^3}\right)^{\frac{1}{2}} + 49\frac{R}{\alpha^3} - 1032\left(\frac{R}{\alpha^3}\right)^{\frac{3}{2}} + \dots\right],$$

which shows that increasing the amplitude of a disturbance of wavelength small compared with the radius of the critical layer increases the rate of damping, but unfortunately does not tell us very much about finite disturbances when α^3/R is small or finite.

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